Internal stabilization of the plate equation in a square: the continuous and the semi-discretized problems.

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Abstract: This paper is devoted to the study of the internal stabilization of the Bernoulli-Euler plate equation in a square. The continuous and the space semi-discretizated problems are successively considered and analyzed using a frequency domain approach. For the infinite dimensional problem, we provide a new proof of the exponential stability result, based on a two dimensional Ingham's type result. In the second and main part of the paper, we propose a finite difference space semi-discretization scheme and we prove that this scheme yields a uniform exponential decay rate (with respect to the mesh size).

Keywords: plate equation, stabilization, uniform exponential stability, finite-difference

AMS subject classification index: 93D15, 65M60, 65M12

1 Introduction and statement of the main results

The aim of this paper is to study the stabilization of the Bernoulli-Euler plate equation from both the theoretical and the numerical points of view. We assume that the domain occupied by the plate is a square and that the plate is subject to a feedback force distributed in a subdomain (internal stabilization). First, we tackle the infinite dimensional problem by using a frequency domain approach, combined to some nonharmonic Fourier analysis results. Then, we propose a finite difference space semi-discretization scheme which yields a uniform exponential decay rate (with respect to the discretization parameter).

The internal or boundary stabilization of the Bernoulli-Euler plate equation has been intensively studied in the literature (see, for instance, [3], [7], [10] and the references therein).

In the case of a rectangular plate with interior control, a sharp result has been proved in Jaffard [7]. This result implies in particular that the plate equation can be exponentially stabilized by means of a feedback acting in an arbitrary subdomain of the rectangle. From the numerical point of view, as far as we know, only the case of one space dimension has been tackled in the literature (see [11]).

This paper can be divided into two parts. The first one contains a new proof of the exponential stability result for the continuous problem. Although this result follows from results proved in [7], we prefer to give a self-contained proof. Instead of using the results in Kahane [9], our argument is based on a new Ingham-Beurling type result of independent interest, which we derive in section 5.1. An additional reason that lead us to propose this new proof lies in the fact that some of the estimates established are useful for the second and main part of this work, devoted to the study of the uniform exponential stability of a space finite difference semi-discretization of the continuous problem. The scheme we propose involves a numerical viscosity term which damps the high frequency modes which cannot be stabilized by the feedback term. The appearance of such spurious modes in the approximation by finite differences or finite elements of control problems has been remarked in several works (see, for instance Glowinski, Li and Lions [4], Infante and Zuazua [5], Tébou and Zuazua [18] and the review paper [20]) proposing various solutions to overcome this difficulty. As far as we know, our results are the first ones concerning the numerical approximation of problems which cannot be handled by multipliers methods. Moreover, our frequency domain approach can be easily adapted to deal with the Schrödinger equation with interior damping.

In order to give the precise statement of our results, we introduce some notation. Consider the square $\Omega = (0,\pi)\times(0,\pi)$ and let $\mathcal{O}\subset\Omega$ be the rectangle $[a,b]\times[c,d]$, with $0<a<b<\pi$ and $0<c<d<\pi$. The set $\mathcal{O}$ represents the part of Ω where the damping is active. Denote by $\chi_{\mathcal{O}}$ the characteristic function of $\mathcal{O}$, and consider the following initial and boundary value problem:

$$\ddot{w}(t) + \Delta^2 w(t) + \chi_{\mathcal{O}}\,\dot{w}(t) = 0, \qquad x\in\Omega,\ t\geqslant 0, \tag{1.1}$$

$$w(t) = \Delta w(t) = 0, \qquad x\in\partial\Omega,\ t\geqslant 0, \tag{1.2}$$

$$w(x,0) = w_0(x), \quad \dot{w}(x,0) = w_1(x), \qquad x\in\Omega, \tag{1.3}$$

where the dot denotes as usual the derivative with respect to time and the last term in the left hand side of (1.1) models the damping effect. The energy of the system at instant t by

$$E(t) = \frac{1}{2}\Big\{||\dot{w}(t)||^2_{L^2(\Omega)} + ||\Delta w(t)||^2_{L^2(\Omega)}\Big\}.$$

Simple formal calculations show that

$$E(t) - E(0) = -\int_0^t\int_{\mathcal{O}} |\dot{w}(s)|^2\,ds, \qquad \forall\ t\geqslant 0,$$

thus the energy is nonincreasing. As already said, the first part of the paper provides a self-contained proof of the following exponential stability result.

Theorem 1.1. *Assume that* $w_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $w_1 \in L^2(\Omega)$. *Then, the system* (1.1)-(1.3) *admits a unique solution* $w \in C(0,\infty; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(0,\infty; L^2(\Omega))$. *Moreover, the system* (1.1)-(1.3) *is exponentially stable, i.e. there exist constants* $M,\ \alpha > 0$ *depending only on* $\mathcal{O}$ *such that*

$$\|\dot{w}(t)\|^2_{L^2(\Omega)} + \|w(t)\|^2_{H^2(\Omega)} \leqslant M\, e^{-\alpha t} \left(\|w_1\|^2_{L^2(\Omega)} + \|w_0\|^2_{H^2(\Omega)} \right) \qquad \forall\, t \geqslant 0.$$

Let us now describe the finite difference space semi-discretization procedure used to approximate the system (1.1)-(1.3). Given $\widetilde{N} \in \mathbb{N}$, we denote by

$$h = \frac{\pi}{\widetilde{N}+1}.$$

Without loss of generality, we can assume that there exist integers $a(h), b(h), c(h), d(h)$ in $\{1, \ldots, \widetilde{N}\}$ such that

$$a = a(h)h, \qquad b = b(h)h, \qquad c = c(h)h, \qquad d = d(h)h. \tag{1.4}$$

Let $w_{j,k}$ denote for all $j, k \in \{0, \widetilde{N}+1\}$ the approximation of the solution w of the system (1.1)-(1.3) at the point $x_{j,k} = (jh, kh)$. We use the standard finite difference approximation of the laplacian

$$\Delta w(jh, kh) \approx \frac{1}{h^2} \left(w_{j+1,k} + w_{j-1,k} + w_{j,k+1} + w_{j,k-1} - 4 w_{j,k} \right) \ \forall j, k \in \{1, \widetilde{N}\}.$$

In order to satisfy the boundary conditions (1.2) we set

$$w_{0,k} = w_{k,0} = w_{\widetilde{N}+1,k} = w_{k,\widetilde{N}+1} = 0 \qquad \forall\, k \in \{0, \ldots, \widetilde{N}+1\},$$

$$w_{-1,k} = -w_{1,k},\ w_{\widetilde{N}+2,k} = -w_{\widetilde{N},k},\ w_{k,-1} = -w_{k,1},\ w_{k,\widetilde{N}+2} = -w_{k,\widetilde{N}} \ \forall k \in \{0, \ldots, \widetilde{N}+1\}.$$

Set

$$V_h = \mathbb{R}^{(\widetilde{N}^2)}.$$

Let $w_h \in V_h$ be the vector whose components are the $w_{j,k}$ for $1 \leqslant j, k \leqslant \widetilde{N}$.

We define the matrix A_{0h} representing the discretization of the bilaplacian with hinged boundary conditions via its square root $A_{0h}^{\frac{1}{2}}$ given by

$$\left(A_{0h}^{\frac{1}{2}} w_h \right)_{j,k} = -\frac{1}{h^2} \left(w_{j+1,k} + w_{j-1,k} + w_{j,k+1} + w_{j,k-1} - 4 w_{j,k} \right),$$

for all $1 \leqslant j, k \leqslant \widetilde{N}$.

The finite-difference scheme for system (1.1)-(1.3) studied in this paper reads then

$$\ddot{w}_{j,k} + (A_{0h} w_h)_{j,k} + (\chi_{\mathcal{O}}\, \dot{w}_h)_{j,k} + h^2 (A_{0h} \dot{w}_h)_{j,k} = 0, \quad 1 \leqslant j, k \leqslant \widetilde{N},\ t \geqslant 0, \tag{1.5}$$

$$w_h(0) = w_{0h}, \qquad \dot{w}_h(0) = w_{1h}. \tag{1.6}$$

In relation (1.5), $\chi_{\mathcal{O}}\,\dot{w}_h$ denotes the vector of V_h whose components are

$$(\chi_{\mathcal{O}}\,\dot{w}_h)_{j,k} = \begin{cases} \dot{w}_{j,k} & \text{if } a(h)\leqslant j\leqslant b(h) \text{ and } c(h)\leqslant k\leqslant d(h),\\ 0 & \text{otherwise.}\end{cases}$$

The numerical viscosity term $h^2 A_{0h}\dot{w}_h$ in (1.5) is introduced in order to damp the high frequency modes. In relation (1.6), w_{0h} and w_{0h} are suitable approximations of the initial data w_0 and w_1 on the finite-difference grid.

The energy of the above semi-discretized system at instant t by

$$E_h(t) = \frac{1}{2}\Big\{\|\dot{w}_h(t)\|^2 + \Big\|A_{0h}^{\frac{1}{2}} w_h(t)\Big\|^2\Big\}.$$

The main result of this paper reads then as follows.

Theorem 1.2. *The family of systems defined by* (1.5)-(1.6) *is uniformly exponentially stable, in the sense that there exist constants C, α, $h^* > 0$ (independent of h, w_{0h} and w_{1h}) such that for all $h \in (0, h^*)$:*

$$\|\dot{w}_h(t)\|^2 + \Big\|A_{0h}^{\frac{1}{2}} w_h(t)\Big\|^2 \leqslant Ce^{-\alpha t}\left(\|w_{1h}\|^2 + \Big\|A_{0h}^{\frac{1}{2}} w_{0h}\Big\|^2\right) \qquad \forall\, t\geqslant 0.$$

The paper is organized as follows. Sections 2 and 3 are respectively devoted to the proofs of theorem 1.1 and theorem 1.2. Some technical results needed in these proofs are given in the appendix constituting section 5.

2 The infinite dimensional problem: proof of theorem 1.1

For all $x \in \mathbb{R}^d$ and all $R\geqslant 0$, we set throughout the paper

$$B(x,R) = \{y \in \mathbb{R}^d,\ |y-x|\leqslant R\},$$

where $|\cdot|$ denotes the Euclidean norm.

In order to establish the exponential stability of the system (1.1)-(1.3) claimed in theorem 1.1, we mainly use three ingredients. The first one is a Hautus type test for the exponential stability of infinite dimensional systems (proposition 2.1). To check that the frequency criterion provided by this Hautus test is satisfied, we use two additional ingredients: an Ingham's type result in $\mathbb{R}^d$ (theorem 2.2) combined to an elementary result from number theory (proposition 5.1).

The first ingredient is a frequency characterization for exponential stability of second order systems. Note that a similar result is proved in [13] for the exact controllability of first order systems.

Proposition 2.1. *Let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint, positive and boundedly invertible operator, with compact resolvent on a Hilbert space H, endowed with the norm $\|\cdot\|$. Let U be another Hilbert space equipped with the norm $\|\cdot\|_U$ and let $B_0 \in \mathcal{L}(U,H)$. Denote by $(\lambda_n)_{n\in\mathbb{N}}$ the increasing sequence formed by the eigenvalues of $A_0^{\frac{1}{2}}$ and let $(\Phi_n)_{n\in\mathbb{N}}$ be a corresponding sequence of eigenvectors, forming an orthonormal basis of H. For all $\omega > 0$ and all $\varepsilon > 0$, define the set*

$$I_\varepsilon(\omega) = \{m \in \mathbb{N} \text{ such that } |\lambda_m - \omega| < \varepsilon\}. \tag{2.1}$$

Then, the following assertions are equivalent:
i) The system

$$\ddot{w}(t) + A_0 w(t) + B_0 B_0^* \dot{w}(t) = 0, \qquad t \geqslant 0, \tag{2.2}$$

$$w(0) = w_0 \in \mathcal{D}(A_0^{\frac{1}{2}}), \quad \dot{w}(0) = w_1 \in H \tag{2.3}$$

is exponentially stable, i.e. there exist positive constants M and α such that

$$\|\dot{w}(t)\|^2 + \|w(t)\|^2_{\mathcal{D}(A_0^{\frac{1}{2}})} \leqslant M\, e^{-\alpha t} \left(\|w_1\|^2 + \|w_0\|^2_{\mathcal{D}(A_0^{\frac{1}{2}})} \right) \qquad \forall\, t \geqslant 0.$$

ii) There exist $\varepsilon > 0$ and $\delta > 0$ such that for all $\omega > 0$ and all $\varphi = \sum_{m\in I_\varepsilon(\omega)} c_m \Phi_m$:

$$\|B_0^* \varphi\|_U \geqslant \delta \|\varphi\|. \tag{2.4}$$

iii) There exist $\varepsilon > 0$ and $\delta > 0$ such that for all $n \in \mathbb{N}^*$ and all $\varphi = \sum_{m\in I_\varepsilon(\lambda_n)} c_m \Phi_m$:

$$\|B_0^* \varphi\|_U \geqslant \delta \|\varphi\|. \tag{2.5}$$

Proof. First, it is clear that conditions *ii*) and *iii*) are equivalent. Indeed, *ii*) obviously implies assertion *iii*) (take $\omega = \lambda_n$). Conversely, if *iii*) holds true for some $\varepsilon > 0$, and if $\omega \in \mathbb{R}$, then either $I_{\varepsilon/2}(\omega)$ is empty, or there exists $n \in I_{\varepsilon/2}(\omega)$ and then $I_{\varepsilon/2}(\omega) \subset I_\varepsilon(\lambda_n)$. Consequently, in both cases, assertion *ii*) holds true.

To prove the equivalence between *i*) and *ii*), we use the exponential stability frequency characterization proved by Liu [12]. According to theorem 3.4. of [12], condition *i*) is equivalent to the assertion

$$\begin{array}{l} \exists \delta > 0 \text{ such that } \forall \varphi \in \mathcal{D}(A_0),\ \forall \omega > 0 : \\ \|(\omega^2 - A_0)\varphi\|^2 + \|\omega\, B_0 B_0^* \varphi\|^2 \geqslant \delta \|\omega \varphi\|^2. \end{array} \tag{2.6}$$

Assume that *ii*) holds true and that (2.6) is false. Then, there exist sequences (ω_n) of $\mathbb{R}_+^*$ and (φ_n) of $D(A_0)$ such that:

$$\|\varphi_n\| = 1, \quad \lim_{n\to+\infty} \left\| \frac{1}{|\omega_n|}(\omega_n^2 - A_0)\varphi_n \right\| = 0, \quad \lim_{n\to+\infty} \|B_0 B_0^* \varphi_n\| = 0. \tag{2.7}$$

Thus, we have

$$\lim_{n\to+\infty} \|B_0^*\varphi_n\|_U^2 = \lim_{n\to+\infty} (B_0 B_0^*\varphi_n, \varphi_n) = 0. \tag{2.8}$$

Decompose $\varphi_n = \sum_{m\in\mathbb{N}} c_m^n \Phi_m$ into

$$\varphi_n = \widetilde{\varphi}_n + \psi_n, \tag{2.9}$$

where

$$\widetilde{\varphi}_n = \sum_{m\in I_n} c_m^n \Phi_m, \tag{2.10}$$

and

$$\psi_n = \sum_{m\notin I_n} c_m^n \Phi_m,$$

with

$$I_n = I_\varepsilon(\omega_n) = \{m \in \mathbb{N} \text{ such that } |\lambda_m - \omega_n| < \varepsilon\},$$

where ε is the constant of assertion ii). Then, we have

$$\left\|\frac{1}{\omega_n}(\omega_n^2 - A_0)\varphi_n\right\|^2 = \sum_{m\in\mathbb{N}} \left|\frac{\omega_n^2-\lambda_m^2}{\omega_n}\right|^2 |c_m^n|^2 \geqslant \varepsilon^2 \sum_{m\notin I_n} \left|\frac{\omega_n+\lambda_m}{\omega_n}\right|^2 |c_m^n|^2 \geqslant \varepsilon^2 \|\psi_n\|^2.$$

The above relation and the second relation of (2.7) imply that

$$\lim_{n\to+\infty} \|\psi_n\| = 0. \tag{2.11}$$

Using relations (2.9), (2.7) and (2.11), we immediately get that

$$\lim_{n\to+\infty} \|\widetilde{\varphi}_n\| = 1. \tag{2.12}$$

Moreover, since $B_0 \in \mathcal{L}(U, H)$, relation (2.11) yields

$$\lim_{n\to+\infty} \|B_0^*\psi_n\|_U = 0.$$

The above relation together with (2.8) show that

$$\lim_{n\to+\infty} \|B_0^*\widetilde{\varphi}_n\|_U = \lim_{n\to+\infty} \|B_0^*\varphi_n - B_0^*\psi_n\|_U = 0.$$

Summing up, the last relation together with relations (2.10) and (2.12) show that ii) is false, and consequently, ii) implies i).
Conversely, assume that i) holds true. Then, (2.6) is satisfied, i.e. there exists $\delta > 0$ such that

$$\|(\omega^2 - A_0)\varphi\|^2 + \|\omega\, B_0 B_0^*\varphi\|^2 \geqslant \delta \|\omega\varphi\|^2, \quad \forall \varphi \in \mathcal{D}(A_0),\ \forall \omega > 0.$$

In particular, for $\omega = \lambda_n$ and $\varphi = \sum_{m\in I_\varepsilon(\lambda_n)} c_m \Phi_m$, we obtain

$$\sum_{m\in I_\varepsilon(\lambda_n)} |\lambda_n^2 - \lambda_m^2|^2 |c_m|^2 + \|\lambda_n\, B_0 B_0^*\varphi\|_U^2 \geqslant \delta \|\lambda_n\varphi\|^2. \tag{2.13}$$

On the other hand, if $m \in I_\varepsilon(\lambda_n)$ and if $\varepsilon < \lambda_1$ then

$$\left|\frac{\lambda_n^2 - \lambda_m^2}{\lambda_n}\right| \leqslant 3\varepsilon.$$

Consequently, by dividing (2.13) by $\lambda_n^2 > 0$, we obtain that for all $\varphi = \sum_{m \in I_\varepsilon(\lambda_n)} c_m \Phi_m$,

$$3\varepsilon\|\varphi\|^2 + \|B_0\|_{\mathcal{L}(U,H)}\|B_0^*\varphi\|_U \geqslant \delta\|\varphi\|^2.$$

By taking $3\varepsilon < \delta$ in the above relation, we get that *iii*) holds. We have already seen that *iii*) and *ii*) are equivalent, thus *ii*) also holds. $\square$

For the proof of theorem 1.1, we also need the following Ingham's type result in $\mathbb{R}^d$, where $d \geqslant 1$ is arbitrary.

Theorem 2.2. *Let $\eta > 0$ and let $(\mu_N)_{N\in\mathbb{N}}$ be a sequence of $\mathbb{R}^d$, where $d \geqslant 1$. For $N \in \mathbb{N}$ and $\gamma > 0$, we set*

$$\kappa_N(\gamma) = \mathrm{Card}\,\{M \in \mathbb{N} \mid \mu_M \in B(\mu_N, \gamma)\}.$$

Assume that the sequence $(\mu_N)_{N\in\mathbb{N}}$ is such that there exist $\gamma > \dfrac{3\sqrt{6}\,d}{\eta}$ and $\gamma' > 0$ satisfying

$$\kappa_N(\gamma') = 1, \qquad \forall\, N \in \mathbb{N}, \tag{2.14}$$

and

$$\kappa_N(\gamma) \leqslant 2, \qquad \forall\, N \in \mathbb{N}. \tag{2.15}$$

Then, there exist $\delta > 0$ depending on γ and γ', but independent of the sequence $(\mu_N)_{N\in\mathbb{N}}$, such that for all set $\mathcal{D} \in \mathbb{R}^d$ containing a ball of radius greater than η and for all sequence $(a_N) \in \ell^2(\mathbb{N}, \mathbb{R})$, the following inequality holds:

$$\int_{\mathcal{D}} \left|\sum_{N\in\mathbb{N}} a_N\, e^{i\,\mu_N\cdot x}\right|^2 \,\mathrm{d}\,x \geqslant \delta \sum_{N\in\mathbb{N}} |a_N|^2. \tag{2.16}$$

In the particular case where $d = 1$, the above result has been proved in [8]. Theorem 2.2 can also be seen as the generalization of the Ingham's type results in $\mathbb{R}^d$ proved in [9] or in Baiocchi, Komornik and Loreti [1]. We remark that the condition $\gamma > \dfrac{3\sqrt{6}\,d}{\eta}$ might be weakened since, for $d = 1$ it has been shown in [1] that a similar result holds for $\gamma > \dfrac{\pi}{\eta}$. However, since this paper deals with stabilizability and not with controllability, a possible improvement in this direction is beyond the scope of this paper. We also remark that, by applying the methodology, introduced in [2] it is easy to check that the conclusion of theorem 2.2 still holds provided that (2.14), (2.15) are satisfied for N large enough.

For the sake of clarity, we postpone the proof of theorem 2.2 to the appendix (see section 5.1).

We are now in position to prove theorem 1.1.

Proof of theorem 1.1. We first note that equations (1.1)-(1.3) can be written in the abstract second order form (2.2)-(2.3) if we introduce the following spaces and operators. Let

$$H = L^2(\Omega), \quad \mathcal{D}(A_0) = \left\{\varphi \in H^4(\Omega) \cap H_0^1(\Omega) \ \mid \Delta\varphi = 0 \text{ on } \partial\Omega \right\}, \tag{2.17}$$

and

$$A_0 : \mathcal{D}(A_0) \to H, \quad A_0\varphi = \Delta^2\varphi \qquad \forall\, \varphi \in \mathcal{D}(A_0). \tag{2.18}$$

If $U = L^2(\mathcal{O})$ denotes the input space, then the input operator $B_0 \in \mathcal{L}(U, H)$ is defined by

$$B_0 u = \widetilde{u}\,\chi_{\mathcal{O}} \qquad \forall\, u \in U = L^2(\mathcal{O}), \tag{2.19}$$

where we have denoted by $\widetilde{u}$ an extension of u to an element of $L^2(\Omega)$. The adjoint of B_0 is clearly given by

$$B_0^*\varphi = \varphi_{|\mathcal{O}} \qquad \forall\, \varphi \in \mathcal{D}(A_0^{\frac{1}{2}}), \tag{2.20}$$

It can be easily checked that A_0 is self-adjoint, positive, boundedly invertible, and that

$$\mathcal{D}\left(A_0^{\frac{1}{2}}\right) = H^2(\Omega) \cap H_0^1(\Omega),$$

with the corresponding norm

$$\|\varphi\|_{\frac{1}{2}}^2 = \int_\Omega |\Delta\varphi(x)|^2 \ \mathrm{d}\,x.$$

Finally, the eigenvalues of $A_0^{\frac{1}{2}}$ are

$$\lambda_{p,q} = p^2 + q^2 \qquad \forall\, p, q \in \mathbb{N}^*. \tag{2.21}$$

A corresponding set of normalized eigenfunctions (in H) is given by

$$\varphi_{p,q}(x) = \frac{2}{\pi}\sin(px_1)\sin(qx_2) \qquad \forall\, p, q \in \mathbb{N}^*, \ \forall\, x = (x_1, x_2) \in \Omega. \tag{2.22}$$

According to proposition 2.1, the system (2.2)-(2.3) is exponentially stable if and only if there exist $\varepsilon, \delta > 0$ such that

$$\forall\, m, n \in \mathbb{N}^*, \ \forall\, \varphi = \sum_{(p,q)\in I_\varepsilon(m,n)} a_{p,q}\varphi_{p,q} : \quad \|B_0^*\varphi\|_U \geqslant \delta\|\varphi\|, \tag{2.23}$$

where

$$I_\varepsilon(m,n) = \{(p,q) \in \mathbb{N}^* \times \mathbb{N}^* \ ; \ |\lambda_{p,q} - \lambda_{m,n}| < \varepsilon\}. \tag{2.24}$$

First, we note that it suffices to check the above condition only for the "high frequencies" (see lemma 2.3 proved below). In other words, it suffices to show that there exist $\delta, r_0 > 0$ such that for all $m, n \in \mathbb{N}^*$ satisfying $r = \sqrt{m^2 + n^2} \geqslant r_0$ and for all $\varphi = \sum_{(p,q)\in I_\varepsilon(m,n)} a_{p,q}\varphi_{p,q}$, we have

$$\|B_0^*\varphi\|_U \geqslant \delta\|\varphi\|. \tag{2.25}$$

Let $\mathcal{C}_r$ denotes the circle with radius $r = \sqrt{m^2+n^2}$ centered at the origin. Since the eigenvalues $\lambda_{p,q}$ of $A_0^{\frac{1}{2}}$ take only integer values, we obviously have for $\varepsilon < 1$

$$I_\varepsilon(m,n) = \{(p,q) \in \mathbb{N}^* \times \mathbb{N}^* \mid p^2+q^2 = m^2+n^2\} = (\mathbb{N}^*)^2 \cap \mathcal{C}_r.$$

Therefore, condition (2.25) is clearly satisfied if there exist $\delta, r_0 > 0$ such that the inequality

$$\int_{\mathcal{O}} \Bigg| \sum_{(p,q)\in(\mathbb{N}^*)^2\cap\mathcal{C}_r} a_{p,q} \sin(px_1)\sin(qx_2) \Bigg|^2 \,\mathrm{d}\,x_1 \,\mathrm{d}\,x_2 \geqslant \delta \sum_{(p,q)\in(\mathbb{N}^*)^2\cap\mathcal{C}_r} |a_{p,q}|^2 \tag{2.26}$$

holds for all $r > r_0$ and for all sequence $(a_{p,q}) \in \ell^2(\mathbb{N}^* \times \mathbb{N}^*, \mathbb{R})$.

In order to establish the above inequality, we first rewrite the sum appearing in the left hand side of (2.26) using complex exponentials. More precisely, using the identity

$$\sin(px_1)\sin(qx_2) = -\frac{1}{4}\Big(e^{i(px_1+qx_2)} + e^{-i(px_1+qx_2)} - e^{i(px_1-qx_2)} - e^{i(-px_1+qx_2)}\Big),$$

we easily obtain that

$$\sum_{(p,q)\in(\mathbb{N}^*)^2\cap\mathcal{C}_r} a_{p,q}\sin(px_1)\sin(qx_2) = -\sum_{(p,q)\in(\mathbb{Z}^*)^2\cap\mathcal{C}_r} \frac{\mathrm{sgn}(p\,q)}{4}\, a_{|p|,|q|}\, e^{i(px_1+qx_2)}.$$

Consequently, to prove that (2.26) holds, it suffices to show that there exist $\delta, r_0 > 0$ such that

$$\int_{\mathcal{O}} \Bigg| \sum_{(p,q)\in(\mathbb{Z}^*)^2\cap\mathcal{C}_r} a_{p,q}\, e^{i(px_1+qx_2)} \Bigg|^2 \,\mathrm{d}\,x_1 \,\mathrm{d}\,x_2 \geqslant \delta \sum_{(p,q)\in(\mathbb{Z}^*)^2\cap\mathcal{C}_r} |a_{p,q}|^2 \tag{2.27}$$

for all $r > r_0$ and for all sequence $(a_{p,q}) \in \ell^2(\mathbb{Z}^* \times \mathbb{Z}^*, \mathbb{C})$.

The main ingredient to prove the above inequality is the Ingham's type result detailed in theorem 2.2. In order to use this result, we need the following notations. Let us denote by $(\mu_1(r), \cdots, \mu_{I(r)}(r))$ the sequence constituted by those points (p,q) of $(\mathbb{Z}^*)^2 \cap \mathcal{C}_r$. Then, proving (2.27) amounts to showing the existence of constants $\delta, r_0 > 0$ such that for all $r > r_0$ and for all complex sequence $(a_1, \cdots, a_{I(r)})$:

$$\int_{\mathcal{O}} \Bigg| \sum_{N=1}^{I(r)} a_N\, e^{i\,\mu_N(r)\cdot x} \Bigg|^2 \,\mathrm{d}\,x \geqslant \delta \sum_{N=1}^{I(r)} |a_N|^2. \tag{2.28}$$

Let $\eta > 0$ be such that $\mathcal{O}$ contains a ball of radius η. According to theorem 2.2, property (2.28) will be satisfied if we can prove the existence of three constants $\gamma > 6\sqrt{6}/\eta$, $\gamma' > 0$ and $r_0 > 0$ such that for all $r > r_0$

$$\kappa_N(\gamma', r) = 1, \qquad \forall\, N = 1, \cdots, I(r) \tag{2.29}$$

and

$$\kappa_N(\gamma, r) \leqslant 2, \qquad \forall\, N = 1, \cdots, I(r) \tag{2.30}$$

where we have set

$$\kappa_N(\gamma, r) = \text{Card}\,\{M \in \mathbb{N}^* \mid 1 \leqslant M \leqslant I(r) \text{ and } \mu_M(r) \in B(\mu_N(r), \gamma)\}, \tag{2.31}$$

for all $N \in \{1, \ldots, I(r)\}$.

Relation (2.29) obviously holds if $\gamma' < 1$. Let $\gamma > 6\sqrt{6}/\eta$. The existence of a constant $r_0 > 0$ such that (2.30) holds for all $r > r_0$ is given by proposition 5.1 (see section 5.2), and the proof of theorem 1.1 is thus complete. □

Lemma 2.3. *Using the notation of the proof above and given $r_0, \varepsilon > 0$, there exists $\delta > 0$ such that for all $(m, n) \in \mathbb{N}^* \times \mathbb{N}^*$ satisfying $0 < \sqrt{m^2 + n^2} < r_0$ and for all $\varphi = \sum_{(p,q) \in I_\varepsilon(m,n)} a_{p,q}\varphi_{p,q}$, we have*

$$\|B_0^* \varphi\|_U^2 \geq \delta \|\varphi\|^2. \tag{2.32}$$

Proof. Given $\varepsilon, r_0 > 0$ and $m, n \in \mathbb{N}^*$ such that $\sqrt{m^2 + n^2} \in (0, r_0)$, we obviously have

$$I_\varepsilon(m, n) \subset I_0 := \{(p, q) \in \mathbb{N}^* \times \mathbb{N}^* \mid \lambda_{p,q} < r_0 + \varepsilon\}.$$

Therefore, the lemma will be proved provided we check (2.32) for all functions $\varphi \in V_0 := \{\varphi = \sum_{(p,q) \in I_0} a_{p,q}\ \varphi_{p,q};\ a_{p,q} \in \mathbb{C}\}$. But since A_0 has compact resolvent, V_0 is finite-dimensional. Combining this argument and the fact that

$$\|B_0^* \varphi\|_U^2 = \int_{\mathcal{O}} |\varphi|^2 > 0, \qquad \forall\, \varphi \in V_0 \setminus \{0\}$$

we obtain the desired conclusion. □

3 A uniformly exponentially stable finite-difference scheme: proof of theorem 1.2

Before proving theorem 1.2, we need some further notation. Recall that

$$V_h = \mathbb{R}^{(\widetilde{N})^2}$$

and let

$$U_h = \mathbb{R}^{(b(h) - a(h) + 1) \times (d(h) - c(h) + 1)}$$

be the discretized input space, where the integers $a(h), b(h), c(h)$ and $d(h)$ are defined by (1.4). We introduce the finite-difference approximation $B_{0h} \in \mathcal{L}(U_h, V_h)$ of the operator B_0 defined by (2.19) by setting for all $u_h \in U_h$

$$(B_{0h}\, u_h)_{j,k} = \begin{cases} u_{j,k} & \text{if } a(h) \leqslant j \leqslant b(h) \text{ and } c(h) \leqslant k \leqslant d(h), \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

The adjoint $B_{0h}^* \in \mathcal{L}(V_h, U_h)$ of B_{0h} is then defined for all $w_h \in V_h$ by

$$\left(B_{0h}^* w_h\right)_{j,k} = w_{j,k}, \qquad a(h)\leqslant j\leqslant b(h) \text{ and } c(h)\leqslant k\leqslant d(h),$$

In the remaining part of this section we denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^m$ for various values of m, and in particular, V_h will be endowed with this norm. The finite-difference semi-discretization(1.5)-(1.6) reads then

$$\ddot{w}_{j,k} + \left(A_{0h} w_h\right)_{j,k} + \left(B_{0h} B_{0h}^* \dot{w}_h\right)_{j,k} + h^2 \left(A_{0h} \dot{w}_h\right)_{j,k} = 0, \qquad 1\leqslant j,k\leqslant \widetilde{N}, \tag{3.2}$$

$$w_{j,k} = \left(A_{0h}^{\frac{1}{2}} w_h\right)_{j,k} = 0, \qquad j,k = 0, \widetilde{N}+1, \tag{3.3}$$

$$w_{j,k}(0) = w_{0h}, \quad \dot{w}_{j,k}(0) = w_{1h}, \quad 0\leqslant j,k\leqslant \widetilde{N}+1. \tag{3.4}$$

It can be easily checked that the sequence $(\|B_{0h}\|_{\mathcal{L}(U_h,V_h)})$ is bounded and that the eigenvalues of $A_{0h}^{\frac{1}{2}}$ are (see for instance [19])

$$\lambda_{p,q,h} = \frac{4}{h^2}\left[\sin^2\left(\frac{ph}{2}\right) + \sin^2\left(\frac{qh}{2}\right)\right], \text{ for } 1\leqslant p,q\leqslant \widetilde{N}. \tag{3.5}$$

A corresponding sequence of normalized eigenvectors in V_h is

$$\varphi_{p,q,h} = \left(\varphi_{p,q,h}^{j,k}\right)_{1\leqslant j,k\leqslant \widetilde{N}}, \quad \varphi_{p,q,h}^{j,k} = \frac{2h}{\pi}\sin(jph)\sin(kqh). \tag{3.6}$$

Note that eigenvectors of the discretized problem are related to those of the continuous one defined in (2.22) by the relation

$$\varphi_{p,q,h}^{j,k} = h\,\varphi_{p,q}(jh, kh). \tag{3.7}$$

The proof of theorem 1.2 is based on the following frequency domain characterization for the uniform exponential stability of a sequence of semigroups (see [14, p.162]).

Theorem 3.1. *Let $(\mathbb{T}_h)_{h>0}$ be a family of semigroups of contractions on the Hilbert space V_h and A_h be the corresponding infinitesimal generators. The family $(\mathbb{T}_h)_{h>0}$ is uniformly exponentially stable if and only if the two following conditions are satisfied:*
i) For all $h>0$, $i\mathbb{R}\subset\rho(A_h)$, where $\rho(A_h)$ denotes the resolvent set of A_h,
ii) $\sup\limits_{h>0,\omega\in\mathbb{R}} \|(i\omega - A_h)^{-1}\| < +\infty$.

Proof of theorem 1.2. In order to apply theorem 3.1, we rewrite the system (3.2)-(3.4) as a first order system. Let us then introduce the space $X_h = V_h \times V_h$, which will be endowed with the norm

$$\|(\varphi_h,\psi_h)\|_{X_h}^2 = \|\varphi_h\|^2 + \left\|A_{0h}^{\frac{1}{2}}\psi_h\right\|^2.$$

Setting $z_h = \begin{bmatrix} w_h \\ \dot{w}_h \end{bmatrix}$, equations (3.2)-(3.4) can be easily written in the equivalent form

$$\dot{z}_h(t) = A_h z_h(t), \quad z_h(0) = z_0$$

where $A_h \in \mathcal{L}(X_h)$ is defined by

$$A_h = \begin{bmatrix} 0 & I \\ -A_{0h} & -h^2 A_{0h} - B_{0h} B_{0h}^* \end{bmatrix}, \quad z_{0h} = \begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix}. \tag{3.8}$$

It will be useful to introduce the operator A_{1h}

$$A_{1h} = \begin{bmatrix} 0 & I \\ -A_{0h} & 0 \end{bmatrix} \in \mathcal{L}(X_h) \tag{3.9}$$

such that

$$A_h = A_{1h} - \begin{bmatrix} 0 & 0 \\ 0 & h^2 A_{0h} + B_{0h} B_{0h}^* \end{bmatrix}. \tag{3.10}$$

We will also need in the sequel the spectral basis of the operator A_{1h}. Moreover, it will be more convenient to number the eigenelements of A_{1h} using only one index m instead of the couple (p, q). To achieve this, let us first rearrange the sequence of eigenvalues $(\lambda_{p,q})_{p,q \in \mathbb{N}^*}$ of the continuous problem and defined by (2.21) in non-decreasing order to obtain a new sequence $(\Lambda_m)_{m \in \mathbb{N}^*}$. Then, if

$$\Lambda_m = \lambda_{p,q} = p^2 + q^2, \qquad \forall\, m \in \mathbb{N}^*,\ \forall p, q \in \mathbb{N}^*, \tag{3.11}$$

then we set for all $1 \leqslant m \leqslant \widetilde{N}^2$, and for all $1 \leqslant p, q \leqslant \widetilde{N}$:

$$\begin{cases} \Lambda_{m,h} = \lambda_{p,q,h}, \\ \varphi_{m,h} = \varphi_{p,q,h}. \end{cases} \tag{3.12}$$

Let then

$$N_2(h) = \widetilde{N}^2 = \left(\frac{\pi}{h} - 1\right)^2$$

be the number of nodes of the finite-difference grid. If we extend the definition of $\Lambda_{m,h}$ and $\varphi_{m,h}$ to the values $m \in \{-1, \ldots, -N_2(h)\}$ by setting

$$\begin{cases} \Lambda_{m,h} = -\Lambda_{-m,h}, \\ \varphi_{m,h} = \varphi_{-m,h}, \end{cases} \tag{3.13}$$

then it can be easily checked that an orthonormal basis of X_h formed by eigenvectors of A_{1h} is given by

$$\Phi_{m,h} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\dfrac{i}{\Lambda_{m,h}} \varphi_{m,h} \\ \varphi_{m,h} \end{bmatrix}, \quad 1 \leqslant |m| \leqslant N_2(h), \tag{3.14}$$

where $\Phi_{m,h}$ is an eigenvector associated to the eigenvalue $i\Lambda_{m,h}$.

We are now in position to apply theorem 3.1. We first check condition $i)$ of theorem 3.1. Suppose that there exist $\begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix} \in X_h$ and $\omega \in \mathbb{R}$ such that

$$A_h \begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix} = i\omega \begin{bmatrix} \varphi_h \\ \psi_h \end{bmatrix}.$$

Then, by using the definition (3.8) of A_h, we easily obtain that

$$\begin{cases} \psi_h = i\omega\varphi_h\,, \\ \Big[\omega^2 - A_{0h} - i\omega(h^2 A_{0h} + B_{0h}B_{0h}^*)\Big]\varphi_h = 0. \end{cases} \tag{3.15}$$

By taking the imaginary part of the inner product of the second relation in (3.15) with φ_h and by using the fact that A_{0h} is invertible, we get that $\varphi_h = 0$. Then, by using the first relation in (3.15) we get that $\psi_h = 0$. Thus, $i\omega$ cannot be an eigenvalue of A_h and hence $i\omega \in \rho(A_h)$ for all $\omega \in \mathbb{R}$. Thus, the spectrum of the operator A_h contains no point on the imaginary axis and condition $i)$ in theorem 3.1 holds true.

To prove condition $ii)$, we use a contradiction argument. Let us thus assume the existence for all $n \in \mathbb{N}$ of $h_n \in (0, h^*)$, $\omega_n \in \mathbb{R}$, $z_n = \begin{bmatrix}\phi_n \\ \psi_n\end{bmatrix} \in X_{h_n}$ such that

$$\|z_n\|^2 = \left\|A_{0h_n}^{\frac{1}{2}}\phi_n\right\|^2 + \|\psi_n\|^2 = 1 \qquad \forall\, n \in \mathbb{N} \tag{3.16}$$

$$\|i\omega_n z_n - A_{h_n} z_n\| \to 0. \tag{3.17}$$

To obtain a contradiction, we decompose z_n into a low frequency part and a high frequency part. More precisely, for $0 < \varepsilon < 1$ and $h \in (0, h^*)$, we define the integer

$$M(h) = \max\Big\{m \in \{1, \ldots, N_2(h)\} \mid h^2(\Lambda_m)^2 \leqslant \varepsilon\Big\}, \tag{3.18}$$

where the sequence $(\Lambda_m)_{m\in\mathbb{N}^*}$ defined in (3.11) constitutes the sequence of eigenvalues of the continuous problem. The eigenvalues $\Lambda_{m,h}$ for $1 \leqslant m \leqslant M_h$ correspond to "low frequencies" and will be damped to zero by the feedback control term $B_{0h}B_{0h}^*\dot{w}_h$. The eigenvalues $\Lambda_{m,h}$ for $m > M_h$ correspond to "high frequencies", and will be damped thanks to the numerical viscosity term.

To get the desired contradiction, we follow several steps.

Step 1
Let us prove the two relations

$$h_n^2 \left\|A_{0h_n}^{\frac{1}{2}}\psi_n\right\|^2 + \|B_{0h_n}^*\psi_n\|^2 \to 0, \tag{3.19}$$

$$\lim_{n\to\infty} \left\|A_{0h_n}^{\frac{1}{2}}\phi_n\right\|^2 = \lim_{n\to\infty} \|\psi_n\|^2 = \frac{1}{2}. \tag{3.20}$$

Relation (3.19) follows directly from (3.17) by taking the inner product in X_{h_n} of $i\omega_n z_n - A_{h_n} z_n$ by z_n and by considering only the real part. By using (3.17), (3.19), (3.10) and the fact that the operators B_{0h_n} are uniformly bounded we obtain that

$$\left\|i\omega_n z_n - A_{1h_n} z_n + \begin{bmatrix} 0 \\ h_n^2 A_{0h_n}\psi_n \end{bmatrix}\right\| \to 0. \tag{3.21}$$

We show now by a contradiction argument that the sequence (ω_n) contains no subsequence converging to zero. Suppose that such a subsequence exists. For the sake of simplicity, we still denote it by (ω_n). Then, by considering the first component of (3.21) and by using (3.16), we obtain that

$$\left\| A_{0h_n}^{\frac{1}{2}} \psi_n \right\| \to 0. \tag{3.22}$$

Moreover, by taking the inner product in V_{h_n} of the second component of

$$i\omega_n z_n - A_{1h_n} z_n + \begin{bmatrix} 0 \\ h_n^2 A_{0h_n} \psi_n \end{bmatrix}$$

by ϕ_n, and by using (3.21), (3.22) and the fact that (h_n) is bounded we get that

$$\left\| A_{0h_n}^{\frac{1}{2}} \phi_n \right\| \to 0.$$

The above relation and (3.22) contradict (3.16), so we have proved that there exists $n_0 \in \mathbb{N}$ such that the sequence $(|\omega_n|)_{n \geqslant n_0}$ is bounded away from zero.
We can now prove (3.20). Taking the inner product in X_{h_n} of (3.21) by $\dfrac{1}{\omega_n} \begin{bmatrix} \phi_n \\ -\psi_n \end{bmatrix}$, with $n \geqslant n_0$, and by considering only the imaginary part, we obtain:

$$\lim_{n \to \infty} \left(\left\| A_{0h_n}^{\frac{1}{2}} \phi_n \right\|^2 - \|\psi_n\|^2 \right) = 0.$$

The above relation and (3.16) yield (3.20). Step 1 is thus complete.

In order to state the second step, let us introduce the modal decomposition of z_n on the spectral basis of $(\Phi_{m,h_n})_{1 \leqslant |m| \leqslant N_2(h_n)}$ of A_{1h_n}. For all $n \in \mathbb{N}$, there exist complex coefficients $(c_m^n)_{1 \leqslant |m| \leqslant N_2(h_n)}$ such that

$$z_n = \begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix} = \sum_{1 \leqslant |m| \leqslant N_2(h_n)} c_m^n \Phi_{m,h_n}. \tag{3.23}$$

The normalization condition (3.16) reads then

$$\sum_{1 \leqslant |m| \leqslant N_2(h_n)} |c_m^n|^2 = 1. \tag{3.24}$$

<u>*Step 2*</u>
In this step, we prove that the following relations holds true

$$\psi_n = \frac{1}{\sqrt{2}} \sum_{m=1}^{N_2(h_n)} \left(c_m^n + c_{-m}^n \right) \varphi_{m,h_n}, \tag{3.25}$$

$$\sum_{M(h_n) < m \leqslant N_2(h_n)} \left| c_m^n + c_{-m}^n \right|^2 \to 0, \tag{3.26}$$

$$\sum_{1\leqslant|m|\leqslant M(h_n)} \left|\omega_n - \Lambda_{m,h_n}\right|^2 \ |c_m^n|^2 \to 0. \tag{3.27}$$

Note that, roughly speaking, relations (3.25) and (3.26) show that the projection of ψ_n on the high frequencies tends to 0 as n tends to $+\infty$.

Relation (3.25) follows directly by taking the second component in (3.23) and by using (3.14).

On the other hand, by using (3.23) and the fact that $\Phi_{m,h}$ is an eigenvector of A_{1h} associated to the eigenvalue $i\Lambda_{m,h}$, we have

$$i\omega_n z_n - A_{1h_n} z_n = \sum_{1\leqslant|m|\leqslant N_2(h_n)} i\left(\omega_n - \Lambda_{m,h_n}\right) c_m^n \Phi_{m,h_n} \tag{3.28}$$

From (3.19) and (3.25) it follows that

$$h_n^2 \left\|A_0^{\frac{1}{2}}\psi_n\right\|^2 = \sum_{m=1}^{N_2(h_n)} h_n^2 \Lambda_{m,h_n}^2 \left|c_m^n + c_{-m}^n\right|^2 \to 0. \tag{3.29}$$

Using the expressions (2.21) and (3.5) of $\lambda_{p,q}$ and $\lambda_{p,q,h}$, it can be easily checked that

$$\frac{4}{\pi^2}\lambda_{p,q} \leqslant \lambda_{p,q,h} \leqslant \lambda_{p,q} \qquad \forall\ 1\leqslant p,q\leqslant \widetilde{N},$$

or equivalently, that

$$\frac{4}{\pi^2}\Lambda_m \leqslant \Lambda_{m,h} \leqslant \Lambda_m \qquad \forall\ 1\leqslant m\leqslant N_2(h). \tag{3.30}$$

Relations (3.29), (3.30) and (3.18) imply (3.26). On the other hand, relations (3.30) and (3.29) clearly imply that there exists a constant C independent of h such that

$$h_n^4 \sum_{m=1}^{M(h_n)} \Lambda_{m,h_n}^4 \left|c_m^n + c_{-m}^n\right|^2 \leqslant C\varepsilon \sum_{m=1}^{M(h_n)} h_n^2 \Lambda_{m,h_n}^2 \left|c_m^n + c_{-m}^n\right|^2 \to 0. \tag{3.31}$$

On the other hand, a simple calculation shows that

$$\begin{bmatrix} 0 \\ h_n^2 A_{0h_n}\psi_n \end{bmatrix} = \sum_{1\leqslant|m|\leqslant N_2(h_n)} \frac{h_n^2}{2}\Lambda_{m,h_n}^2 \left(c_m^n + c_{-m}^n\right) \Phi_{m,h_n}, \tag{3.32}$$

Relations (3.31) and (3.32) imply that

$$\begin{bmatrix} 0 \\ h_n^2 A_{0h_n}\psi_n \end{bmatrix} - \sum_{M(h_n)<|m|\leqslant N_2(h_n)} \frac{h_n^2}{2}\Lambda_{m,h_n}^2 \left(c_m^n + c_{-m}^n\right) \Phi_{m,h_n} \to 0 \tag{3.33}$$

By using (3.21), (3.28) and (3.33) it follows that

$$\sum_{1\leqslant|m|\leqslant N_2(h_n)} i\left(\omega_n - \Lambda_{m,h_n}\right) c_m^n \Phi_{m,h_n} + \sum_{M(h_n)<|m|\leqslant N_2(h_n)} \frac{h_n^2}{2}\Lambda_{m,h_n}^2 \left(c_m^n + c_{-m}^n\right) \Phi_{m,h_n} \to 0.$$

Since the family (Φ_{m,h_n}) is orthogonal, the above relation implies (3.27).

<u>*Step 3*</u>
Consider the set

$$\mathcal{F} = \left\{ n \in \mathbb{N} \mid \exists\, m(n) \in \mathbb{Z},\ 1 \leqslant |m(n)| \leqslant M(h_n),\ \text{such that } |\omega_n - \Lambda_{m(n),h_n}| < \frac{1}{8} \right\}.$$

In other words, $\mathcal{F}$ is constituted by those integers n such that ω_n is located in the "low frequency band". We distinguish then two cases:

First Case. The set $\mathcal{F}$ is finite. Then, for the sake of simplicity, we can suppose, without loss of generality, that $\mathcal{F}$ is empty, i.e., that, for all $n \in \mathbb{N}$, we have:

$$|\omega_n - \Lambda_{m,h_n}| \geqslant \frac{1}{8} \quad \text{for all } 1 \leqslant |m| \leqslant M(h_n).$$

Note that in this case, all the elements of the sequence (ω_n) are located in the "high frequency band".

By using relation (3.27) in Step 2 and the above relation, we obtain that

$$\sum_{1 \leqslant |m| \leqslant M(h_n)} |c_m^n|^2 \to 0,$$

i.e. that the low-frequency part of ψ_n tends to 0. Thus, the above relation, (3.25) and (3.26) in Step 2 imply that

$$\psi_n \to 0 \ \text{ in } \ H,$$

which contradicts (3.20).

Second case. The set $\mathcal{F}$ is infinite. Then, for the sake of simplicity, we can suppose, without loss of generality, that $\mathcal{F} = \mathbb{N}$. In this case, all the sequence ω_n is located in the "low frequency band".

For all $n \in \mathbb{N}$, we introduce the set $\mathcal{F}_n$ defined by

$$\mathcal{F}_n = \left\{ m \in \mathbb{Z} \mid 1 \leqslant |m| \leqslant M(h_n) \text{ and } |\omega_n - \Lambda_{m,h_n}| < \frac{1}{8} \right\}.$$

Note that $\mathcal{F}_n$ is never empty (since it always contains $m(n)$) and represents the collection of low frequency eigenvalues located near ω_n.

Set then

$$\widetilde{\psi}_n = \frac{1}{\sqrt{2}} \sum_{m \in \mathcal{F}_n} c_m^n\, \varphi_{m,h_n}. \tag{3.34}$$

The definition of $\mathcal{F}_n$, together with relation (3.27) of Step 2 imply that

$$\sum_{m \in \{1,\ldots,N_2(h_n)\} \setminus \mathcal{F}_n} |c_m^n|^2 \to 0. \tag{3.35}$$

Using now relations (3.25) and (3.26) of Step 2, we see that (3.35) exactly states that

$$\|\psi_n - \widetilde{\psi}_n\| \to 0. \tag{3.36}$$

The above relation implies (since $(\|B^*_{0h_n}\|)$ is bounded) that

$$\|B^*_{0h_n}(\psi_n - \widetilde{\psi}_n)\| \to 0.$$

This relation together with relation (3.19) of Step 1 show that

$$\|B^*_{0h_n}\widetilde{\psi}_n\| \to 0. \tag{3.37}$$

But on the other hand, we have the following result: there exists $\delta > 0$ such that for all $n \in \mathbb{N}$, we have

$$\|B^*_{0h_n}\widetilde{\psi}_n\|^2 \geqslant \delta^2 \sum_{m\in\mathcal{F}_n} |c^n_m|^2. \tag{3.38}$$

The above relation follows from the application of lemma 3.2 below, if we remark that $\mathcal{F}_n = I_{h_n}(\omega_n)$, where $I_h(\omega)$ is defined for all $h \in (0, h^*)$ and all $\omega \in \mathbb{R}$ in lemma 3.2 .

Gathering (3.35), (3.37) and (3.38), we finally obtain that $\widetilde{\psi}_n \to 0$ in H. By using (3.36), we obtain that $\psi_n \to 0$ which contradicts (3.20). □

A key ingredient in the proof of theorem 1.2 is the following result concerning the observability of a low frequency packet of eigenvectors.

Lemma 3.2. *Let $0 < \varepsilon < 1$, and set for all $h > 0$:*

$$M(h) = \max\Big\{m \in \{1, \dots, N_2(h)\} \mid h^2(\Lambda_m)^2 \leqslant \varepsilon\Big\},$$

where the eiegenvalues (Λ_m) are defined by (3.11). For all $\omega \in \mathbb{R}$, let $I_h(\omega)$ be the set

$$I_h(\omega) = \Big\{m \in \mathbb{Z} \text{ such that } 1 \leqslant |m| \leqslant M(h) \text{ and } |\Lambda_{m,h} - \omega| < \frac{1}{8}\Big\}, \tag{3.39}$$

where the eigenvalues $(\Lambda_{m,h})$ are defined by (3.12) and (3.13).

Then, there exists $h^*, \delta > 0$ such that for all $h \in (0, h^*)$, for all $\omega \in \mathbb{R}$ and for all $\varphi_h = \sum_{m\in I_h(\omega)} c_m\varphi_{m,h}$:

$$\|B^*_{0h}\varphi_h\| \geqslant \delta\|\varphi_h\|. \tag{3.40}$$

Proof. To prove relation (3.40), we are going to use its continuous counterpart (2.23).

Without loss of generality, we can assume that the set $I_h(\omega)$ is included in $\mathbb{N}^*$. It will be more convenient to use the following expression of φ_h:

$$\varphi_h = \sum_{(p,q)\in J_h(\omega)} c_{p,q}\ \varphi_{p,q,h},$$

where the eigenfunctions $\varphi_{p,q,h}$ are defined by (3.6) and where

$$J_h(\omega) = \left\{ (p,q) \in \mathbb{N}^* \times \mathbb{N}^* \mid h^2 \lambda_{p,q}^2 < \varepsilon \text{ and } |\omega - \lambda_{p,q,h}| < \frac{1}{8} \right\} \tag{3.41}$$

is the set described by (p,q) when m describes $I_h(\omega)$ (recall that $\Lambda_m = \lambda_{p,q} = p^2+q^2$).

First of all, let us note that there exists $(p_0, q_0) \in \mathbb{N}^* \times \mathbb{N}^*$ depending only on h such that

$$h^2 \lambda_{p_0,q_0}^2 < \varepsilon \tag{3.42}$$

and satisfying

$$J_h(\omega) \subset \{(p,q) \in \mathbb{N}^* \times \mathbb{N}^* \mid \lambda_{p,q} = \lambda_{p_0,q_0}\}.$$

Indeed, let $(p_0, q_0) \in J_h(\omega)$ (if $J_h(\omega)$ is empty, the claimed result is obvious). Then, for $(p,q) \in J_h(\omega)$, we have

$$|\lambda_{p,q,h} - \lambda_{p_0,q_0,h}| < \frac{1}{4}. \tag{3.43}$$

On the other hand, using the inequality

$$x^2 - \frac{x^4}{3} \leqslant \sin^2(x) \leqslant x^2, \qquad \forall\, x \in \mathbb{R}$$

one easily obtains that

$$\lambda_{p,q} - \frac{\varepsilon}{12} \leqslant \lambda_{p,q,h} \leqslant \lambda_{p,q}, \qquad \forall\, (p,q) \in J_h(\omega). \tag{3.44}$$

Gathering (3.43) and the above estimate, we get (since $\varepsilon < 1$) that

$$|\lambda_{p,q} - \lambda_{p_0,q_0}| \leqslant \frac{1}{2}.$$

Since $\lambda_{p,q}$ takes only integer values, the above relation implies that $\lambda_{p,q} = \lambda_{p_0,q_0}$. Consequently,

$$\begin{aligned} \|B_{0h}^* \varphi_h\|^2 &= \sum_{j=a(h)}^{b(h)} \sum_{k=c(h)}^{d(h)} \left| \sum_{(p,q) \in J_h(\omega)} c_{p,q}\, \varphi_{p,q,h}^{j,k} \right|^2 = \sum_{j=a(h)}^{b(h)} \sum_{k=c(h)}^{d(h)} \left| \sum_{\lambda_{p,q} = \lambda_{p_0,q_0}} c_{p,q}\, \varphi_{p,q,h}^{j,k} \right|^2 \\ &= \sum_{j=a(h)}^{b(h)} \sum_{k=c(h)}^{d(h)} \sum_{\lambda_{p,q} = \lambda_{p_0,q_0}} \sum_{\lambda_{p',q'} = \lambda_{p_0,q_0}} c_{p,q}\, c_{p',q'}\, \varphi_{p,q,h}^{j,k}\, \varphi_{p',q',h}^{j,k} \end{aligned}$$

where we have set $c_{p,q} = 0$ when $\lambda_{p,q} = \lambda_{p_0,q_0}$ and $(p,q) \notin J_h(\omega)$. Note here that the set $\{(p,q) \in \mathbb{N}^* \times \mathbb{N}^* \mid \lambda_{p,q} = \lambda_{p_0,q_0}\}$ depends on h, since (p_0,q_0) does (λ_{p_0,q_0} satisfies (3.42)).

The last equality also reads

$$\|B_{0h}^* \varphi_h\|^2 = \sum_{\lambda_{p,q} = \lambda_{p_0,q_0}} \sum_{\lambda_{p',q'} = \lambda_{p_0,q_0}} c_{p,q}\, c_{p',q'} \left(S_{p,q}^{p',q'}(h) + R_{p,q}^{p',q'}(h) \right) \tag{3.45}$$

where we have set

$$S_{p,q}^{p',q'}(h) = \sum_{j=a(h)}^{b(h)-1} \sum_{k=c(h)}^{d(h)-1} \varphi_{p,q,h}^{j,k}\, \varphi_{p',q',h}^{j,k} \tag{3.46}$$

and

$$R_{p,q}^{p',q'}(h) = \sum_{j=a(h)}^{b(h)} \varphi_{p,q,h}^{j,d(h)}\, \varphi_{p',q',h}^{j,d(h)} + \sum_{k=c(h)}^{d(h)} \varphi_{p,q,h}^{b(h),k}\, \varphi_{p',q',h}^{b(h),k}. \tag{3.47}$$

Let us first deal with the term $S_{p,q}^{p',q'}(h)$. Using equation (3.7) relating the eigenvectors $\varphi_{p,q}$ of A_0 defined by (2.22) to those of A_{0h}, we can rewrite equation (3.46) in the form

$$\begin{aligned} S_{p,q}^{p',q'}(h) &= \sum_{j=a(h)}^{b(h)-1} \sum_{k=c(h)}^{d(h)-1} h^2\, \varphi_{p,q}(jh,kh)\, \varphi_{p',q'}(jh,kh) \\ &= \sum_{j=a(h)}^{b(h)-1} \sum_{k=c(h)}^{d(h)-1} \int_{jh}^{(j+1)h} \int_{kh}^{(k+1)h} \varphi_{p,q}(jh,kh)\, \varphi_{p',q'}(jh,kh)\, \mathrm{d}\,x. \end{aligned} \tag{3.48}$$

Let $\varphi = \sum\limits_{\lambda_{p,q}=\lambda_{p_0,q_0}} c_{p,q}\, \varphi_{p,q}$. Relation (2.23) shows that there exists $\delta_0 > 0$ such that

$$\|B_0^*\varphi\|_{L^2(\mathcal{O})} \geqslant \delta_0 \|\varphi\|_{L^2(\Omega)} = \delta_0 \left(\sum_{\lambda_{p,q}=\lambda_{p_0,q_0}} |c_{p,q}|^2 \right)^{\frac{1}{2}}. \tag{3.49}$$

On the other hand, we have

$$\begin{aligned} \|B_0^*\varphi\|_{L^2(\mathcal{O})}^2 &= \int_a^b \int_c^d \left| \sum_{\lambda_{p,q}=\lambda_{p_0,q_0}} c_{p,q}\, \varphi_{p,q}(x) \right|^2 \mathrm{d}\,x_1\, \mathrm{d}\,x_2 \\ &= \int_a^b \int_c^d \sum_{\lambda_{p,q}=\lambda_{p_0,q_0}} \sum_{\lambda_{p',q'}=\lambda_{p_0,q_0}} c_{p,q} c_{p',q'}\, \varphi_{p,q}(x)\, \varphi_{p',q'}(x)\, \mathrm{d}\,x_1\, \mathrm{d}\,x_2 \\ &= \sum_{\lambda_{p,q}=\lambda_{p_0,q_0}} \sum_{\lambda_{p',q'}=\lambda_{p_0,q_0}} c_{p,q}\; c_{p',q'}\; S_{p,q}^{p',q'} \end{aligned} \tag{3.50}$$

with

$$S_{p,q}^{p',q'} = \sum_{j=a(h)}^{b(h)-1} \sum_{k=c(h)}^{d(h)-1} \int_{jh}^{(j+1)h} \int_{kh}^{(k+1)h} \varphi_{p,q}(x_1,x_2)\, \varphi_{p',q'}(x_1,x_2)\, \mathrm{d}\,x_1\, \mathrm{d}\,x_2.$$

The above relation and (3.46) show that

$$\begin{aligned} |S_{p,q}^{p',q'} - S_{p,q}^{p',q'}(h)| \leqslant \sum_{j=a(h)}^{b(h)-1} \sum_{k=c(h)}^{d(h)-1} \int_{jh}^{(j+1)h} \int_{kh}^{(k+1)h} & |\varphi_{p,q}(x_1,x_2)\, \varphi_{p',q'}(x_1,x_2) \\ & - \varphi_{p,q}(jh,kh)\, \varphi_{p',q'}(jh,kh)|\; \mathrm{d}\,x_1\, \mathrm{d}\,x_2. \end{aligned} \tag{3.51}$$

Using the mean value theorem, we easily obtain that

$$|\varphi_{p,q}(x_1,x_2)\,\varphi_{p',q'}(x_1,x_2)-\varphi_{p,q}(jh,kh)\,\varphi_{p',q'}(jh,kh)|\leqslant C(p+p'+q+q')h \quad (3.52)$$

Since $\lambda_{p,q}=\lambda_{p',q'}=\lambda_{p_0,q_0}$, and since λ_{p_0,q_0} satisfies the low frequency condition (3.42), we have (recall that $\lambda_{p,q}=p^2+q^2$):

$$p+p'+q+q'\leqslant\varepsilon^{\frac{1}{4}}\ h^{-\frac{1}{2}}.$$

Gathering the above relation, (3.51) and (3.52), we obtain that

$$|S_{p,q}^{p',q'}-S_{p,q}^{p',q'}(h)|\leqslant C(b-a)(d-c)\varepsilon^{\frac{1}{4}}\ \sqrt{h}. \quad (3.53)$$

Concerning the term $R_{p,q}^{p',q'}(h)$ defined by (3.47), we easily get that

$$|R_{p,q}^{p',q'}(h)|\leqslant(b-a+d-c)\ h. \quad (3.54)$$

Consequently, using (3.53) and (3.54) in (3.45) and (3.50), we obtain that

$$\begin{aligned}
&\left|\|B_0^*\varphi\|^2_{L^2(\mathcal{O})}-\|B_{0h}^*\varphi_h\|^2\right| \\
&\quad\leqslant \sum_{\lambda_{p,q}=\lambda_{p_0,q_0}}\ \sum_{\lambda_{p',q'}=\lambda_{p_0,q_0}} |c_{p,q}\,c_{p',q'}|\left(\left|S_{p,q}^{p',q'}(h)-S_{p,q}^{p',q'}\right|+\left|R_{p,q}^{p',q'}(h)\right|\right) \\
&\quad\leqslant C\ \sqrt{h}\ \left(\sum_{\lambda_{p,q}=\lambda_{p_0,q_0}}|c_{p,q}|\right)^2 \\
&\quad\leqslant C\ \sqrt{h}\ \kappa(h)\ \left(\sum_{\lambda_{p,q}=\lambda_{p_0,q_0}}|c_{p,q}|^2\right) \quad (3.55)
\end{aligned}$$

where

$$\begin{aligned}
\kappa(h) &= \text{Card }\{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*\mid \lambda_{p,q}=\lambda_{p_0,q_0}\}, \\
&= \text{Card }\{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*\mid p^2+q^2=p_0^2+q_0^2\}.
\end{aligned}$$

Note that in the above definition, $\kappa(h)$ depends on h through p_0 and q_0 (recall that $(p_0,q_0)\in J_h(\omega)$ defined by (3.41)). An estimate of the cardinal $\kappa(h)$ is provided by lemma 3.3 below is

$$\kappa(h)\leqslant C\left(\sqrt{p_0^2+q_0^2}\right)^{\frac{2}{3}}.$$

By using the fact that (p_0,q_0) satisfies the low frequency condition (3.42), the above relation implies that

$$\kappa(h)\leqslant C\ h^{-\frac{1}{3}}.$$

Thanks to relation (3.49), the above inequality and (3.55) yield

$$\|B_{0h}^*\varphi_h\|^2\geqslant((\delta_0)^2-C\,h^{\frac{1}{6}})\ \left(\sum_{\lambda_{p,q}=\lambda_{p_0,q_0}}|c_{p,q}|^2\right).$$

Inequality (3.40) follows from the above relation, by taking h small enough. $\square$

Lemma 3.3. *Given $p_0, q_0 \in \mathbb{N}^*$, let $r = \sqrt{p_0^2 + q_0^2}$. We define*

$$\Theta_r = \{(p,q) \in \mathbb{N}^* \times \mathbb{N}^* \mid p^2 + q^2 = r^2\}.$$

Then, there exists a constant $C > 0$ such that

$$\mathrm{Card}\,(\Theta_r) \leqslant C r^{\frac{2}{3}}.$$

Proof. Denote by $\mathcal{C}_r$ the circle of radius r centered at the origin. Fix $\gamma < r^{\frac{1}{3}}/4$. Then, proposition 5.1 proved below shows that each disk of radius γ centered around an element (p,q) of Θ_r contains at most two elements of Θ_r. The claimed result follows immediately. □

4 Concluding remarks

The result in theorem 1.2 still holds for a rectangular plate. The proof of this fact is essentially the same as in the case of a square plate, with an extra technical difficulty occurring in the proof of the counterpart of Lemma 5.1.

Our frequency domain approach can also be adapted to tackle other numerical dissipation terms. More precisely, if we replace the numerical viscosity $h^2 A_{0h} \dot{w}_h$ in (1.5)-(1.6) by the weaker one $h^2 A_{0h}^{\frac{1}{2}} \dot{w}_h$ we obtain the system

$$\ddot{w}_{j,k} + (A_{0h} w_h)_{j,k} + (\chi_{\mathcal{O}}\, \dot{w}_h)_{j,k} + h^2 \left(A_{0h}^{\frac{1}{2}} \dot{w}_h\right)_{j,k} = 0, \quad 1 \leqslant j,k \leqslant \widetilde{N},\ t \geqslant 0, \tag{4.1}$$

$$w_h(0) = w_{0h}\,, \qquad \dot{w}_h(0) = w_{1h}. \tag{4.2}$$

Recently, Münch and Pazoto in [15] have tackled the corresponding discretization for the wave equation which reads, with our notation,

$$\ddot{w}_{j,k} + \left(A_{0h}^{\frac{1}{2}} w_h\right)_{j,k} + (\chi_{\mathcal{O}}\, \dot{w}_h)_{j,k} + h^2 \left(A_{0h}^{\frac{1}{2}} \dot{w}_h\right)_{j,k} = 0, \quad 1 \leqslant j,k \leqslant \widetilde{N},\ t \geqslant 0, \tag{4.3}$$

$$w_h(0) = w_{0h}\,, \qquad \dot{w}_h(0) = w_{1h}. \tag{4.4}$$

They proved that the solutions of (4.3), (4.4) are uniformly exponentially stable provided that the set $\mathcal{O}$ satisfies a certain geometric optics condition. By using our frequency domain methods it can be checked that the result in [15] implies the uniform exponential stability of (4.1), (4.2). This is essentially due to the fact that the eigenvectors of A_{0h} are the same as those $A_{0h}^{\frac{1}{2}}$ whereas the eigenvalues of A_{0h} are more spaced than those of $A_{0h}^{\frac{1}{2}}$ (see Ramdani, Takahashi, Tenenbaum and Tucsnak [16] for a similar result in a continuous setting).

Moreover, since the results of [15] hold for an arbitrary geometry of Ω (provided that $\mathcal{O}$ satisfies a geometric optics condition), we have that the corresponding semi-discrete problems (4.1), (4.2) are still uniformly exponentially stable in this much more general situation.

Finally, a natural question is the necessity of the numerical viscosity term in order to have uniform exponential stabilization. By following the ideas in [11], it is easy to check that this kind of term is necessary for the corresponding boundary stabilization problem. For the internal stabilization problem considered in the present paper the answer, is not known. However, in the one dimensional case (a hinged plate with internal dissipation) it can be shown that this term is not necessary. Indeed, in this case, the eigenvalues $\lambda_{n,h}$ of the corresponding operator $A_{0h}^{\frac{1}{2}}$ satisfy the uniform gap condition

$$\lambda_{n+1,h} - \lambda_{n,h} \geqslant \gamma, \tag{4.5}$$

for some $\gamma > 0$. Moreover, the corresponding normalized eigenvectors $\varphi_{n,h}$ satisfy the uniform observability estimate

$$\|B_{0h}^* \varphi_{n,h}\| \geqslant \beta > 0. \tag{4.6}$$

The above two estimates can then be combined with a discrete version of Proposition 2.1 to show that the solutions of

$$\ddot{w}_h + A_{0h} w_h + B_{0h} B_{0h}^* \dot{w}_h = 0$$

are uniformly exponentially stable (see [17] for a detailed proof). In the two dimensional case considered in this paper, the observability property (4.6) still holds while the gap condition (4.5) fails, and thus, the method from the 1-d case does not apply. The study of the 2-d case without the numerical viscosity term is therefore an open question.

5 Appendix

5.1 An Ingham's type inequality in $\mathbb{R}^d$: Proof of theorem 2.2

The proof of theorem 2.2 follows the idea of Ingham (cf. [6]) and can be seen as a generalization of the proof given in [8] for real valued sequences $(\mu_N)_{N\in\mathbb{N}}$ to sequences with values in $\mathbb{R}^d$, where d is arbitrary. In order to present the main idea of the proof, let us consider a sequence $(a_N)_{N\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{C})$ and $k \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. If we introduce the Fourier transform K of k defined by

$$K(\lambda) = \int_{\mathbb{R}^d} k(x) e^{i\lambda\cdot x} \, \mathrm{d}\, x, \tag{5.1}$$

then we clearly have

$$\int_{\mathbb{R}^d} k(x) \left| \sum_{N\in\mathbb{N}} a_N \, e^{i\,\mu_N\cdot x} \right|^2 \mathrm{d}\, x = \sum_{N\in\mathbb{N}} \sum_{M\in\mathbb{N}} a_N \, \overline{a_M} \, K(\mu_N - \mu_M). \tag{5.2}$$

Moreover, if k satisfies $k(x) \leqslant 0$ for all $|x| \geqslant \eta$, for some $\eta > 0$, then (5.2) implies that

$$\sum_{N\in\mathbb{N}} \sum_{M\in\mathbb{N}} a_N \, \overline{a_M} \, K(\mu_N - \mu_M) \leqslant \|k\|_{L^\infty(\mathbb{R}^d)} \int_{|x|\leqslant\eta} \left| \sum_{N\in\mathbb{N}} a_N \, e^{i\,\mu_N\cdot x} \right|^2 \mathrm{d}\, x. \tag{5.3}$$

We are thus lead to estimate the left hand side of the above relation. The key point is to choose the function K such that the diagonal terms of this left hand side dominate the non diagonal ones. To achieve this, let us admit for the moment the existence of a function $k \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that the following assumptions hold (recall that K is given by (5.1)):

$$K \text{ is non negative and even,} \tag{5.4}$$

$$k(x) \leqslant 0, \qquad \forall\, |x| \geqslant \eta, \tag{5.5}$$

$$\text{the support of } K \text{ is contained in } B(0,\gamma), \tag{5.6}$$

$$K \text{ admits a strict maximum at } 0, \tag{5.7}$$

The sum $\displaystyle\sum_{N\in\mathbb{N}} \sum_{M\in\mathbb{N}} a_N\, \overline{a_M}\, K(\mu_N - \mu_M)$ can be written in the form $S_1 + S_2$ where

$$S_1 = \sum_{\kappa_N(\gamma)=1} \sum_{M\in\mathbb{N}} a_N\, \overline{a_M}\, K(\mu_N - \mu_M), \tag{5.8}$$

and

$$S_2 = \sum_{\kappa_N(\gamma)=2} \sum_{M\in\mathbb{N}} a_N\, \overline{a_M}\, K(\mu_N - \mu_M). \tag{5.9}$$

On the one hand, by using (5.6), we have

$$S_1 = K(0) \sum_{\kappa_N(\gamma)=1} |a_N|^2. \tag{5.10}$$

For S_2, we note that if N satisfies $\kappa_N(\gamma) = 2$, then the set $\{M \in \mathbb{N} \mid \mu_M \in B(\mu_N, \gamma)\}$ contains exactly two elements, namely N and N'. Thus, by using (5.4) and (5.6), we have

$$2S_2 = \sum_{\kappa_N(\gamma)=2} \left(|a_N|^2 + |a_{N'}|^2\right) K(0) + 2\, \mathrm{Re}(a_N\, \overline{a_{N'}})\, K(\mu_N - \mu_{N'}). \tag{5.11}$$

Moreover, since $|\mu_N - \mu_{N'}| \geqslant \gamma'$, assumption (5.7) shows the existence of a constant $\alpha \in (0,1)$ such that

$$K(\mu_N - \mu_{N'}) \leqslant \alpha K(0).$$

Combining the above relation and (5.11) yields

$$S_2 \geqslant (1-\alpha) K(0) \sum_{\kappa_N(\gamma)=2} |a_N|^2.$$

The above relation and (5.10) yields

$$\sum_{N\in\mathbb{N}} \sum_{M\in\mathbb{N}} a_N\, \overline{a_M}\, K(\mu_N - \mu_M) = S_1 + S_2 \geqslant (1-\alpha) K(0) \sum_{N\in\mathbb{N}} |a_N|^2. \tag{5.12}$$

The desired conclusion (2.16) of theorem 2.2 follows then from (5.12) and (5.3), with

$$\delta = \frac{(1-\alpha)K(0)}{\|k\|_{L^\infty(\mathbb{R}^d)}}.$$

To achieve the proof, it remains to prove the existence of $k \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ satisfying (5.4)-(5.7) (recall that K is the Fourier transform of k, and is defined by (5.1)).

Let χ be the characteristic function of $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and set

$$g = \chi * \chi * \chi * \chi * \chi * \chi.$$

We define

$$G(\lambda_1, \cdots, \lambda_d) = \prod_{i=1}^{d} g(\lambda_i), \tag{5.13}$$

and

$$K_0 = G + \frac{1}{6d}\Delta G. \tag{5.14}$$

Let

$$k_0(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} K_0(\lambda) e^{-i\lambda\cdot X} \,\mathrm{d}\,\lambda$$

denote the inverse Fourier transform of K_0.

Obviously, we have for all $x = (x_1, \cdots, x_d)$:

$$k_0(x) = \frac{1}{(2\pi)^d}\left(1 - \frac{|x|^2}{6d}\right)\prod_{i=1}^{d}\left(\frac{\sin(x_i/2)}{x_i}\right)^6.$$

Therefore, k_0 is negative outside $B(0, \sqrt{6d})$. Moreover, the function K_0 is even and its support is contained in $[-3,3]^d \subset B(0, 3\sqrt{d})$. Let us show that K_0 is non negative and admits a strict maximum at 0. Using (5.13) and (5.14), we get that

$$K_0(\lambda_1, \cdots, \lambda_d) = \frac{1}{d}\sum_{i=1}^{d}\left(\prod_{j\neq i} g(\lambda_j)\right)\left(g(\lambda_i) + \frac{1}{6}g''(\lambda_i)\right).$$

It has been shown in [8] that the functions g and $g + \frac{1}{6}g''$ are non negative and admit a strict maximum at 0 (in reference [8], the function g is denoted χ^5). Consequently, K_0 is non negative and admits a strict maximum at 0. To conclude the proof, we note that if we set

$$K(\lambda) = K_0\left(\frac{3\sqrt{d}}{\gamma}\lambda\right)$$

then

$$k(x) = \left(\frac{\gamma}{3\sqrt{d}}\right)^d k_0\left(\frac{\gamma}{3\sqrt{d}}\,x\right)$$

and conditions (5.4)-(5.7) are fulfilled, where η in (5.5) satisfies $\eta > \dfrac{3\sqrt{6}\,d}{\gamma}$. The proof is thus complete.

5.2 On the asymptotic distribution of the eigenvalues of the Dirichlet Laplacian operator in a square

The result proved in this section constitutes a crucial ingredient of the proofs of our two main results, namely theorem 1.1 and theorem 1.2 (through lemma 3.3). This result provides a very useful piece of information on the asymptotic distribution of the eigenvalues (p^2+q^2), with $p,q\in\mathbb{N}^*$, of the Dirichlet Laplacian operator on the square $(0,\pi)\times(0,\pi)$.

Before giving the precise statement of this result, let us give a formal version of it. Consider the set S_r constituted of the eigenvalues located on a circle of radius $r=\sqrt{m^2+n^2}>0$, with $m,n\in\mathbb{N}^*$, centered at the origin. Given $\gamma>0$, the following proposition shows that, provided r is chosen large enough, then around each element (p^*,q^*) of S_r, there is at most one other element of S_r which is contained in the ball of radius $\gamma>0$ centered at (p^*,q^*). More precisely, we have

Proposition 5.1. *Given $m,n\in\mathbb{N}^*$, let $\mathcal{C}_r$ denotes the circle of radius $r=\sqrt{m^2+n^2}$ centered at the origin. We denote by $(\mu_1(r),\cdots,\mu_{I(r)}(r))$ the sequence constituted by the points (p,q) of $(\mathbb{Z}^*)^2\cap\mathcal{C}_r$. Finally, for $\gamma>0$, define for all $1\leqslant N\leqslant I(r)$:*

$$\kappa_N(\gamma,r)=\mathrm{Card}\,\{M\in\mathbb{N}\mid 1\leqslant M\leqslant I(r) \text{ and } \mu_M(r)\in B(\mu_N(r),\gamma)\} \tag{5.15}$$

where $B(\mu_N(r),\gamma)$ denotes the ball of center $\mu_N(r)$ with radius γ. Then, for all $\gamma>0$ and for all $r>r_0=\max(1,(4\gamma)^3)$, we have

$$\kappa_N(\gamma,r)\leqslant 2, \qquad \forall\, N=1,\cdots,I(r). \tag{5.16}$$

An important ingredient of the proof of proposition 5.1 is lemma 5.2 below, for which we need the following additional notation. Given $r>0$, set (see Figure 1)

$$C_1=\left\{(p,q)\in\mathbb{Z}^2|\ p^2+q^2=r^2,\quad q>0,\quad |p|<\frac{\sqrt{3}r}{2}\right\},\quad C_2=-C_1,$$

$$C_3=\left\{(p,q)\in\mathbb{Z}^2|\ p^2+q^2=r^2,\quad p>0,\quad |q|<\frac{\sqrt{3}r}{2}\right\},\quad C_4=-C_3,$$

and let $d:\mathbb{R}^2\to\mathbb{R}_+$ denote the Euclidean distance in $\mathbb{R}^2$.

Lemma 5.2. *Assume that A_1 A_2, A_3 are three mutually distinct points of C_i for some $i\in\{1,2,3,4\}$. Then*

$$d(A_1,A_2)+d(A_2,A_3)\geqslant\frac{\sqrt[3]{r}}{2}.$$

Proof. It clearly suffices to show that the result holds for $A_1,\ A_2,\ A_3\in C_1$. Denote by f the function $f:[-r,r]\to\mathbb{R}$ defined by $f(t)=\sqrt{r^2-t^2}$. Without loss of generality we can assume that the coordinates of $A_1,\ A_2$ and A_3 are respectively

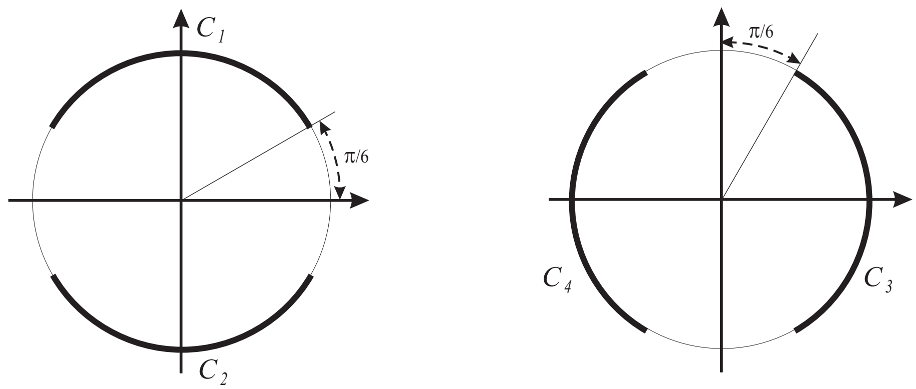

Figure 1: The sets C_1, C_2, C_3, C_4.

$(p, f(p))$, $(p+h, f(p+h))$ and $(p+h+k, f(p+h+k))$, with $p,\ h,\ k \in \mathbb{N}^*$. By applying the mean value theorem there exist $\xi_1 \in (p, p+h)$ and $\xi_2 \in (p+h, p+h+k)$ such that

$$f'(\xi_1) = \frac{N_1}{h}, \quad f'(\xi_2) = \frac{N_2}{k}.$$

where $N_1 = f(p+h) - f(p)$ and $f(p+h+k) - f(p+h)$ are two integers. The same mean value theorem applied to f' yields the existence $\xi \in (\xi_1, \xi_2)$ and of a non vanishing integer N such that

$$f''(\xi) = \frac{N}{(\xi_1 - \xi_2)hk}.$$

The above inequality combined to the fact that $|f''(t)| \leqslant \frac{8}{r}$ for $t \in \left[-\frac{\sqrt{3}}{2}r, \frac{\sqrt{3}}{2}r\right]$ imply that

$$(h+k)^3 \geqslant \frac{r}{8}.$$

The above inequality clearly implies the conclusion of the lemma. □

Proof of proposition 5.1. We use a contradiction argument. Assume that there exist $\gamma > 0$ and

$$r > r_0 = \max(1, (4\gamma)^3) \tag{5.17}$$

such that

$$\kappa_N(\gamma, r) \geqslant 3,$$

where $\kappa_N(\gamma, r)$ is defined by (5.15). In other words, we assume that the set

$$\mathcal{C}_r \cap (\mathbb{Z}^*)^2 = \{(p,q) \in (\mathbb{Z}^*)^2 \mid p^2 + q^2 = r^2\}$$

contains three mutually distinct points $\{A_1, A_2, A_3\}$ satisfying $A_1, A_3 \in B(A_2, \gamma)$. Without loss of generality, we can then assume that $A_1 \in C_1$, that the smallest arc connecting A_1 and A_3 is clockwise and that it contains A_2 (cf. Figure 2).

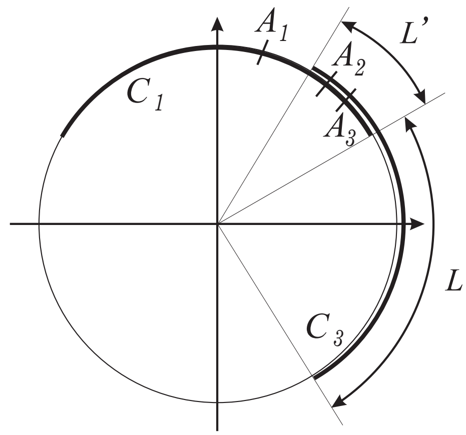

Figure 2: The points A_1, A_2, A_3.

If $A_3 \in C_1$, lemma 5.2 can be applied and shows that $2\gamma \geqslant \dfrac{\sqrt[3]{r}}{2}$, which contradicts (5.17). Assume now that $A_3 \notin C_1$. We remark that the angle $\theta_{13} = \widehat{A_3 O A_1}$ satisfies

$$0 \leqslant \sin(\theta_{13}/2) = \frac{d(A_1, A_3)}{2r} \leqslant \frac{\gamma}{r} \leqslant \frac{1}{4\, r^{2/3}}.$$

Since $r \geqslant 1$, the above expression implies that $\theta_{13}/2 \in [0, \pi/2]$, and consequently, the length L_{13} of the smallest arc connecting A_1 and A_3 satisfies

$$L_{13} = r\theta_{13} \leqslant 2r\, \frac{2}{\pi} \sin(\theta_{13}/2) \leqslant \frac{1}{\pi}\, r^{1/3}.$$

Let $L' = \dfrac{\pi}{2} r$ and $L' = \dfrac{\pi}{6} r$ denote the lengths of the arcs shown in Figure 2. One can easily check that for $r \geqslant 1$, we have $L_{13} < L$ and that $L_{13} < L'$. These two properties imply respectively that $A_3 \in C_3$ and that $A_1 \in C_3$. Lemma 5.2 applies then and provides as above the desired contradiction. $\square$

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